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ORTHOGONAL ALMOST LOCALLY MINIMAL PROJECTIONS ON ℓ_1^n

BY

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ABSTRACT

A projection P on a Banach space X with $\|P\| \le \lambda_0$ is called almost locally minimal if, for every $\alpha > 0$ small enough, the ball $B(P, \alpha)$ in the space of operators $L(X)$ does not contain a projection Q with $\|Q\| \le \|P\|(1 - D\alpha^2)$, where $D = D(\lambda_0)$ is a constant independent of $\|P\|$. It is shown that, for every $p \ge 1$ and every compact abelian group G, every translation invariant projection on $L_p(G)$ is almost locally minimal. Orthogonal projections on ℓ_1^n are investigated with respect to some weaker local minimality properties.

1. Introduction

The purpose of this paper is to study projections on some classical Banach spaces which have the following local minimality property.

Definition 1.1: Let $\lambda_0 > 1$ and $D = 10 + 12\lambda_0^2$. A projection P on a Banach space X with $\|P\| = \lambda < \lambda_0$ is called **almost locally minimal** (a.l.m. for short) if, for every $0 < \alpha < (8\lambda_0)^{-1}$, the ball $B(P, \alpha)$ in the space of operators $L(X)$ does not contain a projection Q with $\|Q\| \le \lambda(1 - D\alpha^2)$.

Almost locally minimal projections were defined and characterized in [Z-1] in the case of finite dimensional spaces as follows:

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PROPOSITION 1.2 (Theorem 2.4 of [Z-1]): *A projection P on a finite dimensional space X is almost locally minimal if, and only if, there is an operator $S \in L(X)$ satisfying the following three conditions:* $\mathrm{tr} SP = \|P\|$, *the nuclear norm* $\|S\|_\wedge = 1$ *and* $SP = PS$.

Let us start by making a few remarks on a.l.m. projections. Definition 1.1 determines a certain guaranteed rate of a norm reduction for a given projection P, if it is not a.l.m., in the following sense: for some $\alpha > 0$, at a distance $\leq \alpha$ there is a projection Q with $\|Q\| \sim \|P\| - \alpha^2$. On the other hand, an a.l.m. projection P has the following minimality property:

Remark 1.3: Let P_0 be an a.l.m. projection on a space X and let P_1 be another projection on X with $(P_0 - P_1)^2 = 0$ (a condition which holds, in particular, when $P_1(X) = P_0(X)$ or $P_1^*(X^*) = P_0^*(X^*)$ but also in many other cases). Then $\|P_1\| \geq \|P_0\|$.

Proof: Let $0 < t < 1$ and put $P_t = (1-t)P_0 + tP_1$. Then $P_t^2 - P_t = -(1-t)t(P_0 - P_1)^2$ so P_t is a projection for some $0 < t < 1$ if and only if $(P_0 - P_1)^2 = 0$. Suppose that $\|P_1\|\|P_0\|^{-1} = \mu < 1$; then $\|P_0 - P_t\| = t\|P_0 - P_1\|$ while $\|P_t\| \leq (1-t)\|P_0\| + t\|P_1\| = [(1-t) + t\mu]\|P_0\| = \|P_0\|[1 - (1-\mu)t]$. Let $\alpha = t\|P_0 - P_1\|$; then we get that $\|P_0 - P_t\| = \alpha$ while $\|P_t\| < \|P_0\|[1 - D\alpha^2]$ if t is small enough, contradicting the fact that P_0 is a.l.m. ∎

COROLLARY 1.4: *An a.l.m. projection P has the minimal norm among all projections Q which can be connected to P by a line segment in $L(X)$ consisting of projections. The argument in Remark 1.3 shows that this set of projections is identical with* $\pi(P) = \{Q \in L(X) \colon Q^2 = Q \text{ and } (Q-P)^2 = 0\}$.

COROLLARY 1.5: (a) *There exists no a.l.m. projection P of norm $\|P\| > 1$ in Hilbert space, because the orthogonal projection Q onto $P(X)$ has norm 1.* (b) *There exist no a.l.m. projections of rank 1 with norm $\|P\| > 1$ because onto every one-dimensional subspace there exists a projection Q with $\|Q\| = 1$.*

The purpose of this paper is to study natural examples of a.l.m. projections on the classical spaces (Sections 2 and 3). In Sections 4 and 5 we pay special attention to the space ℓ_1^n and to "special" projections on this space which have weaker local minimality properties. This study leads to a characterization of a.l.m. projections on ℓ_1^n which is slightly different from Proposition 1.2 (see Theorem 6.6). We conclude with open problems and remarks in Section 6.

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2.1; without his contribution I would be left with only one infinite sequence of examples of a.l.m. projections.

2. Examples of almost locally minimal projections

It is very hard to use Definition 1.1 in order to check if a projection is a.l.m. and, in view of Corollary 1.5, it may seem that there are not too many a.l.m. projections around. The following results show that the classical spaces have many a.l.m. projections.

THEOREM 2.1: *Let G be a finite abelian group and let P be a translation invariant projection on $L_p(G)$. Then P is a.l.m.*

Proof: We will use Proposition 1.2. Let $X = L_p(G)$, suppose that $\|P\| = \lambda$ and choose $\sigma = L(X)^*$ such that $\|\sigma\| = 1$ and $\sigma(P) = \lambda$. The functional $\sigma \in L(X)^*$ can be represented by an operator S on X via the relation $\sigma(T) = \mathrm{tr} ST$ for all $T \in L(X)$ where, as is well known, $1 = \|\sigma\| = \|S\|_\wedge$. Using Rudin's averaging procedure [R], let $\hat{S} = \int_G T_{g^{-1}} S T_g dg$ where dg represents the Haar measure on G and T_g denotes the translation by g. We claim that the translation invariant operator $\hat{S}$ satisfies the three conditions of Proposition 1.2. First of all, since P commutes with translations, we have that

$$\mathrm{tr}(P\hat{S}) = \int_G \mathrm{tr}(PT_{g^{-1}}ST_g)dg = \int_G \mathrm{tr}(T_{g^{-1}}PST_g)dg = \int_G \mathrm{tr}(PS)dg = \mathrm{tr}(PS) = \lambda.$$

Secondly, because $\|S\|_\wedge = 1$ and $\|T_g\|_p = 1$ for all $g \in G$ and $1 \leq p \leq \infty$, we get that $\|\hat{S}\|_\wedge \leq 1$. Finally, since both P and $\hat{S}$ are translation invariant, they commute. By Proposition 1.2, P is a.l.m. ∎

A close look at the proof of Proposition 1.2 above (Theorem 2.4 of [Z-1]) leads to the following extension of Theorem 2.1.

THEOREM 2.2: *Let G be a compact abelian group, let $\lambda_0 > 1$ and let P be a bounded, translation invariant projection on $L_p(G)$ with $\|P\| = \lambda \leq \lambda_0$. Then P is a.l.m.*

Proof: Suppose that P is not a.l.m. and $1 < \|P\| = \lambda < \lambda_0$. Then there is an α, $(8\lambda_0)^{-1} > \alpha > 0$, and a projection Q on $L_p(G)$ with $\|P - Q\| \leq \alpha$ and $\|Q\| \leq \lambda(1 - D\alpha^2)$. The proof of the sufficiency part of Theorem 2.4 of [Z-1] provides us with two automorphisms U and V on $L_p(G)$ such that the following conditions hold,

$$\begin{aligned} &\|U\| \leq (1 - \lambda\alpha)^{-1}, \quad \|V\| \leq (1 - \lambda\alpha)^{-1} \quad \text{and} \\ &\|QU(I - Q)\| \leq 4\alpha\lambda(1 + \lambda) \end{aligned} \tag{2.1}$$

and

$$(2.2)\quad P-Q=-\,[(I-P)VP+(Q-P)U(P-Q)-PU(I-P) \\ +(Q-P)QU(I-Q)+(Q-P)QU(Q-P)+PQU(I-P)].$$

For the precise definitions of U and V and their properties see the proof of Theorem 5.2 below. Select $e\in L_p(G)$ and $f^*\in L_p(G)^*$ with $\|e\|=1=\|f^*\|$ such that $\|Pe\|\geq\lambda(1-(1/2)\alpha^2)$ and $f^*(Pe)\geq\lambda(1-\alpha^2)$ and consider the operator $S=f^*\otimes e$ on $L_p(G)$. Clearly, $\mathrm{tr}PS=f^*(Pe)\geq\lambda(1-\alpha^2)$ and $\|S\|_\wedge\leq 1$. The operators P and S may not commute, but let $\hat{S}=\int_G T_{g^{-1}}ST_g dg$; then, as in the proof of Theorem 2.1 above, we have

$$(2.3)\qquad \mathrm{tr}(P\hat{S})\geq\lambda(1-\alpha^2),\quad \|\hat{S}\|_\wedge\leq 1\quad\text{and}\quad \hat{S}P=P\hat{S}.$$

Because $\hat{S}P=P\hat{S}$, $\hat{S}=P\hat{S}P+(I-P)\hat{S}(I-P)$. Therefore, $\mathrm{tr}(\hat{S}(I-P)VP)=0=\mathrm{tr}(\hat{S}P(U-QU)(I-P))$. It follows from (2.1), (2.2) and (2.3) that

$$\begin{aligned}\lambda(1-\alpha^2)-\lambda(1-D\alpha^2)\leq&\lambda(1-\alpha^2)-\|Q\|\\ \leq&\mathrm{tr}(\hat{S}(P-Q))\\ =&-\,[\mathrm{tr}(\hat{S}(Q-P)U(P-Q))\\ &+\mathrm{tr}(\hat{S}(Q-P)QU(I-Q))\\ &+\mathrm{tr}(S(Q-P)QU(Q-P))]\\ \leq&(1-\lambda\alpha)^{-1}\alpha^2+4\alpha^2\lambda(1+\lambda)+\lambda(1-\lambda\alpha)^{-1}\alpha^2.\end{aligned}$$

Hence

$$\lambda(D-1)\alpha^2\leq[2\lambda(1-\lambda\alpha)^{-1}+4\lambda(1+\lambda)]\alpha^2,$$

an absurdity, by the definition of D (Definition 1.1). ∎

Remark 2.3: Walter Rudin's averaging procedure ([R] Theorem 1), which is used above, showed that a bounded translation invariant projection P on $L_p(G)$ has minimal norm relative to all projections onto the **same** translation invariant subspace. Corollary 1.4 and Theorem 2.2 extend this minimality property to the set $\pi(P)$. Moreover, Theorem 2.2 replaces this minimality property by the almost local minimality property relative to **all close-by** projections Q on $L_p(G)$.

3. Orthogonal almost locally minimal projections on ℓ_1^n

Identifying operators T on ℓ_1^n with the matrix $(t_{i,j})_{i,j=1}^n$ which represents them, with respect to the unit vector basis $\{u_i\}_{i=1}^n$, we call a projection $P = (p_{i,j})$ on ℓ_1^n **orthogonal** if P is an orthogonal projection on ℓ_2^n, i.e. $p_{j,i} = \bar{p}_{i,j}$ for all $1 \leq i,j \leq n$. A representation theorem (Theorem 3.6 of [Z-1]) for a.l.m. projections on ℓ_1^n takes a very useful form in the special case of orthogonal projections, namely,

PROPOSITION 3.1: *Let P be an orthogonal a.l.m. projection on ℓ_1^n with $\|P\| = \lambda > 1$. Then there is an $2 \leq m \leq n$ and a permutation $\{u_i\}_{i=1}^n$ of the unit vector basis of ℓ_1^n with respect to which*

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

where P_1 and P_2 are projections with $\|P_1\| = \lambda$, $\|P_2\| \leq \lambda$, P_1 is an $m \times m$ matrix, $\sum_{i=1}^m |p_{i,j}| = \lambda$ for all $1 \leq j \leq m$ and there exist positive numbers $\{\lambda_i\}_{i=1}^m$ with $\sum_{i=1}^m \lambda_i = 1$ such that $\lambda_i^{-1} \sum_{j=1}^m \lambda_j |p_{i,j}| = \lambda$ for all $1 \leq i \leq m$.

Remark 3.2: A square matrix P_1 with the above properties is said to be **equivalent to a λ-doubly stochastic projection**. In the case $\lambda_i = m^{-1}$, $\sum_{j=1}^m |p_{i,j}| = \lambda$ for all $1 \leq i \leq m$ and we call P_1 **λ-doubly stochastic** (λ-d.s. for short). If an operator T is λ-doubly stochastic then $\|T\| = \lambda$ in both the $L(\ell_1^n)$ and the $L(\ell_\infty^n)$ norms. If T is equivalent to a λ-d.s. operator, then, by [Z-1], in the case of rational λ_i's, there is an isometric embedding J of ℓ_1^n into some larger ℓ_1^N so that T is canonically transformed to a λ-d.s. operator on ℓ_1^N. It easily follows from Proposition 1.2 that, under such an isometric embedding, an a.l.m. projection on ℓ_1^n is transformed to an a.l.m. projection with the same norm and isometrically identical range because there is a canonical norm-1 projection of ℓ_1^N onto $J(\ell_1^n)$.

In view of Theorem 2.1, if G is a finite abelian group then every translation invariant projection P on $L_1(G) = \ell_1^N$, being orthogonal, has the representation suggested by Proposition 3.1. The question, what are the matrices P_1 and P_2 of the above representation theorem in this case, is answered in the following:

PROPOSITION 3.3: *Let G be a finite abelian group and let P be a translation invariant projection on $L_1(G)$ with $\|P\| = \lambda$. Let $|G| = N$ and let $(p_{i,j})$ be the $N \times N$ matrix representing P with respect to the unit vector basis $\{u_g\}_{g \in G}$ of $\ell_1^N = L_1(G)$, where $u_g = \|1_g\|^{-1} 1_g$. Then the matrix $(p_{i,j})$ is λ-doubly stochastic.*

Proof: Since P is translation invariant, for every g, $h \in G$, the set $\{p_{j,g}\colon j \in G\}$ of the components of the vector Pu_g is identical with the set $\{p_{j,h}\colon j \in G\}$ of the components of Pu_h. Therefore, for every $g \in G$, $\lambda = \|Pu_g\| = \sum_{j\in G} |p_{j,g}|$. Because P is an orthogonal projection, $\sum_{g\in G} |p_{j,g}| = \lambda$ for all $j \in G$ and hence P is λ-doubly stochastic. In the notation of Proposition 3.1, $P = P_1$ and $P_2 = 0$. ∎

4. Orthogonally almost locally minimal projections on ℓ_1^n

We start with a weaker property of local minimality for **orthogonal** projections on ℓ_1^n.

Definition 4.1: Let $\lambda_0 > 1$ and $D = 10 + 12\lambda_0^2$. An orthogonal projection P on ℓ_1^n with $\|P\| = \lambda < \lambda_0$ is called orthogonally almost locally minimal (o.a.l.m. for short) if for every $\alpha > 0$ small enough, the ball $B(P, \alpha)$ contains no orthogonal projection Q with $\|Q\| \le \lambda(1 - D\alpha^2)$.

Notation: Given $T = (t_{i,j}) \in L(\ell_1^n)$ we denote by $T^\#$ the operator on ℓ_1^n represented by the matrix $(t^\#_{i,j}) = (\overline{t_{j,i}})$. The purpose of this section is to prove the following:

THEOREM 4.2: *An orthogonal projection P on ℓ_1^n is o.a.l.m. if and only if there is an operator $S \in L(\ell_1^n)$ satisfying the following three conditions:* $\mathrm{tr}(SP) = \|P\|$, $\|S\|_\wedge = 1$ *and* $(S + S^\#)P = P(S + S^\#)$.

We need three preliminary lemmas for the proof of Theorem 4.2. In the proof we need to construct projections with certain properties. The first tool is devised to construct a projection out of an operator which behaves in a fashion similar to that of a projection.

LEMMA 4.3 ([Z-2]): *Let $\lambda_0 > 1$ and let X be a Banach space. There exist a constant $C = C(\lambda_0)$ and a continuous function $\beta(T)$, defined for all operators T on X which satisfy the conditions $\|T\| \le \lambda_0$ and $\|T^2 - T\| = \alpha \le \frac{1}{8}$, such that $\beta(T)$ is a projection and $\|\beta(T) - T\| \le C\alpha$. Moreover, if T is hermitian so is $\beta(T)$.*

Remark 4.4: The detailed proof of Lemma 4.3 is given in [Z-2]. It does not address the hermitian case formally, but an easy examination shows that, because it uses an iterative process which preserves the hermitian structure, the proof yields the hermitian case too.

The second tool we need was used in the proof of Theorem 2.3 of [Z-1] and we will prove it here for completeness. It will tell us how to obtain operators T satisfying the condition of Lemma 4.3.

LEMMA 4.4: *Let P be a projection on a Banach space X and let W be an operator with $\|W\| = 1$ for which*

$$W = PW(I-P) + (I-P)WP.$$

Then, for every $\delta > 0$,

$$\|(P-\delta W)^2 - (P-\delta W)\| \le \delta^2.$$

Proof:

$$\begin{aligned}\|(P-\delta W)^2 - (P-\delta W)\| =&\|P - \delta PW - \delta WP + \delta^2 W^2 - P + \delta W\| \\ =&\|P - \delta PW(I-P) - \delta(I-P)WP + \delta^2 W^2 - P \\ &+ \delta PW(I-P) + \delta(I-P)WP\| \\ \le&\delta^2. \quad \blacksquare\end{aligned}$$

The third preliminary lemma describes a norm reduction operation of an ℓ_1^n vector.

LEMMA 4.5: *Let $\lambda > 1$ and $1 > \gamma_0 > 0$ be constants and assume that $p = (p_1, \dots, p_n)$ and $w = (w_1, \dots, w_n)$ are vectors with $\sum_{h=1}^n |p_h| = \lambda$ and $\sum_{h=1}^n |w_h| \le 1$. Let $A = \{h\colon p_h = 0\}$, $\gamma_h = e^{-i\alpha(h)}$ with $\alpha(h) = \arg p_h$ for those $h \notin A$ and $\gamma_h = -e^{-i\beta(h)}$ with $\beta(h) = \arg w_h$ for $h \in A$.*

If $\mathbf{Re}(\sum_{h=1}^n \gamma_h w_h) \ge \gamma_0$ then there is a δ_0 such that, for every $0 < \delta < \delta_0$, $\|p - \delta w\|_1 \le \lambda - \frac{1}{2}\gamma_0\delta$.

Proof: Let $c = \max\{|p_h|^{-1}|w_h| : h \notin A\}$ and, by continuity, choose $\delta_0 > 0$ so small that $\mathbf{Re}\sum_{h=1}^n \gamma_h' w_h > \frac{1}{2}\gamma_0$ whenever $\max\{|\gamma_h' - \gamma_h| : 1 \le h \le n\} < 2\delta_0 c(1-\delta_0 c)^{-1}$. Let $0 < \delta < \delta_0$ and put $\theta(h) = \arg(p_h - \delta w_h)$ for $h \notin A$. By the triangle inequality, if $h \notin A$ then

$$\begin{aligned}|e^{-i\theta(h)} - e^{-i\alpha(h)}| =&\left|(p_h - \delta w_h)^{-1}|p_h - \delta w_h| - p_h^{-1}|p_h|\right| \\ =&(|p_h - \delta w_h||p_h|)^{-1}\Big[\left||p_h - \delta w_h|p_h - |p_h|p_h\right| \\ &+ \left||p_h|p_h - (p_h - \delta w_h)|p_h|\right|\Big] \\ \le&2(|p_h - \delta w_h||p_h|)^{-1}|p_h|\delta|w_h| \\ =&2\delta|p_h|^{-1}|w_h|(1 - \delta|p_h|^{-1}|w_h|)^{-1} \\ \le&2\delta c(1-\delta c)^{-1} < 2\delta_0 c(1-\delta_0 c)^{-1}.\end{aligned}$$

Let $\gamma'_h = e^{-i\theta(h)}$ if $h \notin A$ and $\gamma'_h = \gamma_h$ for $h \in A$. The choice of δ_0 ensures that, whenever $0 < \delta < \delta_0$,

$$\sum_{h=1}^{n} |p_n - \delta w_h| = \mathbf{Re} \sum_{h=1}^{n} \gamma'_h (p_h - \delta w_h) \leq \sum_{h=1}^{n} |p_h| - \mathbf{Re}\delta \sum_{h=1}^{n} \gamma'_h w_h$$
$$\leq \lambda - \frac{1}{2}\gamma_0 \delta. \qquad ∎$$

Proof of Theorem 4.2: Let $(p_{i,j})$ be the $n \times n$ matrix which represents P with respect to the unit vector basis $\{u_i\}_{i=1}^n$ of ℓ_1^n. Suppose that $\|P\| = \lambda$; then, we may assume w.l.o.g. that there is an $1 \leq m \leq n$ such that $\sum_{i=1}^n |p_{i,j}| = \lambda$ for $1 \leq j \leq m$ while $\sup\{\sum_{i=1}^n |p_{i,j}| \colon m < j \leq n\} = \mu < \lambda$. Let Φ denote the set of all m-tuples $\varphi = (\varphi_1, \ldots, \varphi_m)$ of non-negative numbers $\{\varphi_i\}_{i=1}^m$ with $\sum_{i=1}^m \varphi_i = 1$. Let Γ denote the set of all m-tuples of functionals $g = (g_1, g_2, \ldots, g_m)$ with $\|g_j\|_\infty = 1$ such that $g_j(\sum_{i=1}^n p_{i,j} u_i) = \lambda$ for all $1 \leq j \leq m$. Note that if $g_j = \sum_{h=1}^n g_{j,h} u_h^*$ (where $\{u_h^*\}_{h=1}^n$ is the unit vector basis of ℓ_∞^n), then $g_{j,h} = e^{-i\theta}$ if $p_{h,j} = re^{i\theta} \neq 0$ while in the case $p_{h,j} = 0$, $g_{j,h}$ may be any number with $|g_{j,h}| \leq 1$. Define the operator $S(\varphi, g)$ by $S(\varphi, g) = \sum_{i=1}^m \varphi_i g_i \otimes u_i$ and let

$$\Delta = \{S(\varphi, g) \colon \varphi \in \Phi \ \text{ and } \ g \in \Gamma\}.$$

Clearly, Δ is a compact convex set in $L(\ell_1^n)$. Let $K = \{T \in L(\ell_1^n) \colon T^\# = -T\}$, let $\mathrm{Com}P$ denote the commutant of the projection P and put $\Omega = \mathrm{Com}P + K$. There are two possibilities: either (a) $\Omega \cap \Delta \neq \emptyset$ or (b) $\Omega \cap \Delta = \emptyset$. In case (a), there exists in Δ an operator $S = S(\varphi, g) = T + V$ where $T \in \mathrm{Com}P$ and $V^\# = -V$. Because P is an orthogonal projection and $PT = TP$, we also have that

$$PT^\# = (TP^\#)^\# = (TP)^\# = (PT)^\# = T^\# P^\# = T^\# P.$$

It follows that

$$\begin{aligned}(S + S^\#)P &= (T + V + T^\# + V^\#)P \\ &= (T + T^\#)P = P(T + T^\#) = P(T + V + T^\# + V^\#) \\ &= P(S + S^\#).\end{aligned}$$

Clearly, $\|S\|_\wedge \leq \sum_{j=1}^m \varphi_j \|g_j\| \|u_j\| = 1$ and

$$\mathrm{tr}(PS) = \sum_{j=1}^m \varphi_j g_j(Pu_j) = \sum_{j=1}^m \varphi_j \lambda = \lambda.$$

This establishes the statement of Theorem 4.2 in case (a).

Now suppose that (b) holds. Because $\Delta \cap \Omega = \emptyset$, Δ is compact and convex and because Ω is a subspace of $L(\ell_1^n)$, there is a separating functional W^* on $L(\ell_1^n)$ and a positive γ_0 such that $W^*(T) = 0$ for all $T \in \Omega$ and $\mathbf{Re}W^*(T) > \gamma_0$ for all $T \in \Delta$. Let the operator W represent W^*; then we have that

$$\text{tr}WT = 0 \quad \text{for all } T \in \Omega \tag{4.1}$$

and

$$\mathbf{Re}(\text{tr}WT) \geq \gamma_0 \quad \text{for all } T \in \Delta. \tag{4.2}$$

We may assume w.l.o.g. that $\|W\| = 1$ and put $W = (w_{h,j})$. Let us explain the meaning of (4.1). Picking any $h \neq j$, $1 \leq h, j \leq n$, and putting $w_{h,j} = re^{-i\theta}$, let T denote the matrix for which $(T)_{h,j} = e^{-i\theta}$, $(T)_{j,h} = -e^{i\theta}$ and $(T)_{p,q} = 0$ for all $(p,q) \neq (h,j)$. We get that $T^{\#} = -T$, hence $T \in \Omega$ and therefore $0 = \text{tr}(TW) = w_{j,h}e^{-i\theta} - w_{h,j}e^{i\theta}$ and so $w_{j,h} = w_{h,j}e^{2i\theta} = \overline{w_{h,j}}$. It follows that W is hermitian.

We also have that for every $T \in L(\ell_1^n)$, $PTP + (I-P)T(I-P) \in \text{Com}P \subset \Omega$ and hence $\text{tr}(W(PTP + (I-P)T(I-P))) = 0$, but substituting $W = PWP + (I-P)W(I-P) + PW(I-P) + (I-P)WP$ we get that, for every $T \in L(\ell_1^n)$, $0 = \text{tr}((PWP + (I-P)W(I-P))T)$; therefore $0 = PWP + (I-P)W(I-P)$ and we are left with

$$W = PW(I-P) + (I-P)WP. \tag{4.3}$$

This is the hypothesis of Lemma 4.4. We proceed now to explain the significance of (4.2). For every $1 \leq j \leq m$ let $g_j = (g_{j,1}, \dots, g_{j,n}) \in \ell_\infty^n$ be the functional on ℓ_1^n defined, for every $1 \leq j \leq m$, by

$$\begin{aligned} g_{j,h} &= e^{i\alpha} \quad \text{if } p_{h,j} \neq 0 \text{ and } \alpha = -\arg p_{h,j}, \\ g_{j,h} &= -e^{i\beta} \quad \text{if } p_{h,j} = 0 \text{ and } \beta = -\arg w_{h,j}. \end{aligned} \tag{4.4}$$

Clearly, the operator $g_j \otimes u_j \in \Delta$ and hence

$$\begin{aligned} 0 < \gamma_0 \leq \mathbf{Re}(\text{tr}(Wg_j \otimes u_j)) &= \mathbf{Re}(g_j(Wu_j)) \\ &= \mathbf{Re}(\sum_{h=1}^{n} g_{j,h}w_{h,j}) \quad \text{for } 1 \leq j \leq m. \end{aligned} \tag{4.5}$$

It follows from Lemma 4.5 (with $\gamma_n = g_{j,h}$, $w_h = w_{h,j}$ and $p_h = p_{h,j}$) that if $\delta > 0$ is small enough, then, for every $1 \le j \le m$, $\|Pu_j - \delta \sum_{h=1}^n w_{h,j} u_h\| \le \lambda - \frac{1}{2}\gamma_0 \delta$. Since $\|W\| = 1$, and $\max\{\sum_{h=1}^n |p_{h,j}| : m < j \le n\} = \mu < \lambda$, by choosing $\delta_0 < \frac{1}{2}(\lambda - \mu)$, we get that

$$(4.6) \qquad \|P - \delta W\| \le \|P\| - \frac{1}{2}\gamma_0 \delta \quad \text{for all } 0 < \delta < \delta_0.$$

In view of (4.3), the operator $T = P - \delta W$ satisfies, by Lemma 4.4, the inequality $\|T^2 - T\| \le \delta^2$. Hence, by Lemma 4.3, there is an orthogonal projection $Q = \beta(T)$ on ℓ_1^n such that $C\delta^2 \ge \|Q - T\| = \|Q - (P - \delta W)\|$. It follows from (4.6) that $\|Q\| \le \|P - \delta W\| + C\delta^2 \le \lambda - \frac{1}{2}\gamma_0\delta + C\delta^2$, while $\|P - Q\| \le \|P - T\| + \|T - Q\| \le \|\delta W\| + C\delta^2$. For small values of δ this contradicts the assumption that P is o.a.l.m. ∎

5. Universally bounded projections on ℓ_1^n

Orthogonal projections with norm λ and λ-d.s. projections P on ℓ_1^n share the property that their norms, as members of both $L(\ell_1^n)$ and $L(\ell_\infty^n)$, are $\|P\|_1 = \|P\|_\infty = \lambda$. By the classical interpolation theorem, $\|P\|_p \le \lambda$ for all $1 \le p \le \infty$. Let us define, for any $n \times n$ matrix T,

$$\|T\|_0 = \max\{\|T\|_1, \|T\|_\infty\} = \max\{\|T\|_1, \|T^\#\|_1\}$$

and call $\|\cdot\|_0$ **the universal norm**. Clearly, for an orthogonal or λ-d.s. projection P, $\|P\|_1 = \|P\|_0$.

Definition 5.1: A projection P on ℓ_1^n with $\|P\|_0 = \lambda < \lambda_0$ is called universally almost locally minimal (u.a.l.m. for short) if, for every $\alpha > 0$ small enough, the ball $B_0(P, \alpha) = \{T \in L(\ell_1^n) : \|P - T\|_0 \le \alpha\}$ does not contain a projection Q with $\|Q\|_0 \le \lambda(1 - D\alpha^2)$.

Since the set of orthogonal projections Q on ℓ_1^n with $\|Q\|_1 \le \lambda$ is much smaller than the set of all projections Q with $\|Q\|_0 \le \lambda$, every orthogonal projection P on ℓ_1^n which is u.a.l.m. is obviously o.a.l.m. It turns out that the converse is also true.

THEOREM 5.2: *Let P be an orthogonal o.a.l.m. projection on ℓ_1^n with $\|P\|_1 = \lambda > 1$. Then P is u.a.l.m.*

Proof: By Theorem 4.2, there exists an operator S on ℓ_1^n such that

$$\|S\|_\wedge = 1, \quad \mathrm{tr} SP = \lambda \quad \text{and} \quad (S + S^\#)P = P(S + S^\#).$$

Let $0 < \alpha < [8\lambda(1+\lambda)]^{-1}$ and suppose that Q is a projection on ℓ_1^n with $\|P-Q\|_0 \le \alpha$ and $\|Q\|_0 \le \lambda(1-D\alpha^2)$. In order to prove the theorem it suffices to construct operators A and $\{B_i\}_{i=1}^6$ on ℓ_1^n such that

$$P - \frac{1}{2}(Q+Q^{\#}) = A + \sum_{i=1}^{6} B_i, \tag{5.1}$$

where

$$A^{\#} = A, \quad A = (I-P)T_1P + PT_2(I-P) \quad \text{for some } \{T_i\}_{i=1}^2 \in L(\ell_1^n),$$

and where $\sum_{i=1}^6 \|B_i\|_1 \le (2+12\lambda^2)\alpha^2$.

Indeed, once (5.1) is established, because $S+S^{\#}$ commutes with P, $S+S^{\#} = PT_3P + (I-P)T_3(I-P)$ for some $T_3 \in L(\ell_1^n)$ and so $\operatorname{tr}((S+S^{\#})A) = 0$. Moreover, because $A = A^{\#}$ we get

$$\begin{aligned} \lambda D\alpha^2 &= \lambda - \lambda(1-D\alpha^2) \le \lambda - \frac{1}{2}(\|Q\|_1 + \|Q^{\#}\|_1) \\ &= \lambda - \frac{1}{2}(\|Q\|_1\|S\|_\wedge + \|Q^{\#}\|_1\|S\|_\wedge) \\ &\le \operatorname{tr}(SP) - \frac{1}{2}[\operatorname{tr}(SQ) + \operatorname{tr}(SQ^{\#})] \\ &= \operatorname{tr}(S[P - \frac{1}{2}(Q+Q^{\#})]) = \operatorname{tr}(SA) + \sum_{i=1}^6 \operatorname{tr}(SB_i) \\ &= \frac{1}{2}\operatorname{tr}(S(A+A^{\#})) + \sum_{i=1}^6 \operatorname{tr}(SB_i) \\ &= \frac{1}{2}\operatorname{tr}(SA + S^{\#}A)) + \sum_{i=1}^6 \operatorname{tr}(SB_i) \\ &= \frac{1}{2}\operatorname{tr}((S+S^{\#})A) + \sum_{i=1}^6 \operatorname{tr}(SB_i) \\ &\le (2+12\lambda^2)\alpha^2. \end{aligned} \tag{5.2}$$

The inequality (5.2) is absurd in view of the size of D. It thus remains to prove (5.1).

The first part of the construction is identical with the first part of the proof of Theorem 2.4 of [Z-1]. We will repeat the argument for the sake of completeness. Let $T = PQ + I - Q$; then T maps $(I-Q)(\ell_1^n)$ identically onto itself and $Q(\ell_1^n)$ into $P(\ell_1^n)$. Moreover, $\|I-T\|_0 = \|(Q-P)Q\|_0 \le \lambda\alpha$ hence T is invertible,

$T(Q(\ell_1^n)) = P(\ell_1^n)$ and, if $V = T^{-1}$, then $\|V\|_0 \le (1-\lambda\alpha)^{-1}$. We claim that the operator $R = P + (I-P)VP$ is a projection of ℓ_1^n onto $Q(\ell_1^n)$ along $(I-P)(\ell_1^n)$. Indeed, $R^2 = R$, $I - R = (I-P)(I-VP)$ and, by the definition of T, for every $y \in Q(\ell_1^n)$, $Ty = Py$. Therefore

$$Ry = Py + (I-P)T^{-1}Py = Py + (I-P)T^{-1}Ty = y.$$

Hence R is a projection of ℓ_1^n onto $Q(\ell_1^n)$ with kernel $(I-P)(\ell_1^n)$. Moreover, if $\|x\| \le 1$, then $y = T^{-1}Px \in Q(\ell_1^n)$ and $\|y\| \le (1-\lambda\alpha)^{-1}\lambda\|x\| \le \lambda(1-\lambda\alpha)^{-1}$ and so

(5.3)
$$\begin{aligned}\|Rx - Px\| &= \|(I-P)T^{-1}Px\| = \|(I-P)Qy\| \\ &= \|(Q-P)Qy\| \le \alpha\lambda\|y\| \le \alpha\lambda^2(1-\lambda\alpha)^{-1} \le 2\lambda^2\alpha.\end{aligned}$$

Replacing P and Q by $I-Q$ and $I-P$, respectively, in the first argument and putting $W = (I-Q)(I-P) + P$ we get that $W|_{P(\ell_1^n)}$ is the identity on $P(\ell_1^n)$ and

$$\|I - W\|_0 = \|Q(I-P)\|_0 = \|Q(Q-P)\|_0 \le \lambda\alpha.$$

Hence, W is invertible and it maps $(I-P)(\ell_1^n)$ isomorphically onto $(I-Q)(\ell_1^n)$. We let $U = W^{-1}$, obtain $\|U\|_0 \le (1-\lambda\alpha)^{-1}$ and consider $\tilde{R} = I - Q + QU(I-Q)$. Then $\tilde{R}$ is a projection with kernel $Q(\ell_1^n)$ and, if $y \in (I-P)(\ell_1^n)$, then $Wy = (I-Q)y$. Therefore

$$\tilde{R}y = (I-Q)y + QU(I-Q)y = y.$$

It follows that $\tilde{R}$ is a projection of ℓ_1^n onto $(I-P)(\ell_1^n)$ with $\ker \tilde{R} = Q(\ell_1^n)$ and $\tilde{R} = I - R$. Hence $I - Q + QU(I-Q) = I - P - (I-P)VP$ and

$$P - Q = -[(I-P)VP + QU(I-Q)]. \tag{5.4}$$

Moreover, for every $x \in \mathrm{Ball}(\ell_1^n)$, if $y = W^{-1}(I-Q)x$, then $y \in (I-P)(\ell_1^n)$, $\|y\| \le (1-\lambda\alpha)^{-1}(1+\lambda)$ and

$$\begin{aligned}\|QU(I-Q)x\|_1 &= \|Q(I-P)y\|_1 = \|Q(Q-P)y\|_1 \\ &\le \lambda\alpha\|y\|_1 \le \lambda\alpha(1+\lambda)(1-\lambda\alpha)^{-1}\|x\|_1 \le 4\alpha\lambda(1+\lambda)\|x\|_1.\end{aligned}$$

A similar computation, using the ℓ_∞^n norm, yields $\|QU(I-Q)\|_\infty \le 4\alpha\lambda(1+\lambda)$, therefore

$$\|QU(I-Q)\|_0 \le 4\alpha\lambda(1+\lambda). \tag{5.5}$$

Taking the transpose of (5.4) we get that

$$P - Q^{\#} = -[PV^{\#}(I-P) + (I-Q)^{\#}U^{\#}Q^{\#}].$$

It follows that

$$(5.6)\quad P - \frac{1}{2}(Q+Q^{\#}) = -\frac{1}{2}\big[(I-P)VP + PV^{\#}(I-P) + QU(I-Q) + (I-Q^{\#})U^{\#}Q^{\#}\big].$$

Let us find more manageable expressions for $QU(I-Q)$ and $(I-Q^{\#})U^{\#}Q^{\#}$.

First note that

$$(5.7)\quad \begin{aligned}(Q-P)U(P-Q) &= (Q-P)U(I-Q) - (Q-P)U(I-P)\\ &= QU(I-Q) - PU(I-Q) - QU(I-P) + PU(I-P)\\ &= QU(I-Q) - QU(I-P) + PU(I-P)\end{aligned}$$

(because U maps $(I-Q)(\ell_1^n)$ onto $(I-P)(\ell_1^n)$, yielding $PU(I-Q)=0$). Also,

$$(5.8)\quad \begin{aligned}QU(I-P) &= Q^2U(I-P)\\ &= (Q-P)QU(I-P) + PQU(I-P)\\ &= (Q-P)QU(I-Q) + (Q-P)QU(Q-P) + PQU(I-P).\end{aligned}$$

Combining (5.7) and (5.8) we get that

$$(5.9)\quad \begin{aligned}QU(I-Q) =&(Q-P)U(P-Q) + QU(I-P) - PU(I-P)\\ =&(Q-P)U(P-Q) + (Q-P)QU(I-Q)\\ &+ (Q-P)QU(Q-P) + PQU(I-P) - PU(I-P).\end{aligned}$$

By taking the transpose of both sides we arrive at

$$(5.10)\quad \begin{aligned}(I-Q^{\#})U^{\#}Q^{\#} =&(P-Q^{\#})U^{\#}(Q^{\#}-P)\\ &+ (I-Q^{\#})U^{\#}Q^{\#}(Q^{\#}-P)\\ &+ (Q^{\#}-P)U^{\#}Q^{\#}(Q^{\#}-P)\\ &+ (I-P)U^{\#}Q^{\#}P - (I-P)U^{\#}P.\end{aligned}$$

Hence, by (5.6), (5.9) and (5.10) we have

(5.11)
$$\begin{aligned}P-\frac{1}{2}(Q+Q^{\#}) = &-\frac{1}{2}\big[(I-P)(V-U^{\#}+U^{\#}Q^{\#})P\\ &+P(V^{\#}-U+QU)(I-P)\big]-\frac{1}{2}\big[(Q-P)U(P-Q)\\ &+(P-Q^{\#})U^{\#}(Q^{\#}-P)+(Q-P)QU(I-Q)\\ &+(I-Q^{\#})U^{\#}Q^{\#}(Q^{\#}-P)+(Q-P)QU(Q-P)\\ &+(Q^{\#}-P)U^{\#}Q^{\#}(Q^{\#}-P)\big].\end{aligned}$$

Put $A=(I-P)T_1P+PT_2(I-P)$ where $T_1=-\frac{1}{2}(V-U^{\#}+U^{\#}Q^{\#})$ and $T_2=T_1^{\#}$. Then, clearly, A satisfies the requirements of (5.1). Put $B_1=-\frac{1}{2}(Q-P)U(P-Q)$ and $B_2=B_1^{\#}$; then, since $\|Q-P\|_0\le\alpha$, both $\|B_1\|_1$ and $\|B_2\|_1$ do not exceed $\frac{1}{2}\alpha^2\|U\|_0\le\frac{1}{2}(1-\lambda\alpha)^{-1}\alpha^2\le\alpha^2$. Let $B_3=-\frac{1}{2}(Q-P)QU(I-Q)$ and $B_4=B_3^{\#}$. Since $\|Q-P\|_0=\|Q^{\#}-P\|_0\le\alpha$ and, by (5.5), $\|QU(I-Q)\|_0\le 4\alpha\lambda(1+\lambda)$, we get that $\|B_3\|_1$ and $\|B_4\|_1$ do not exceed $2\alpha^2\lambda(1+\lambda)$. Finally, put $B_5=-\frac{1}{2}(Q-P)QU(Q-P)$ and $B_6=B_5^{\#}$. Then, because $\|Q-P\|_0\le\alpha$ we see that $\|B_5\|_1$ and $\|B_6\|_1$ do not exceed $\alpha^2\lambda(1-\lambda\alpha)^{-1}\le 2\lambda\alpha^2$. It follows that $\sum_{i=1}^6\|B_i\|_1\le\alpha^2(2+4\lambda(1+\lambda)+4\lambda)<\alpha^2(2+12\lambda^2)$. This establishes (5.1) and completes the proof of Theorem 5.2. ∎

6. Remarks and open problems

The isomorphic type of the range of a projection on ℓ_1^n is far from being understood. In view of the above results it may be useful to examine the ranges of some special projections. In the sequel, $\varphi(\lambda)$ denotes a function which depends on λ but is independent of n.

PROBLEM 6.1: *Does there exist a function $\varphi(\lambda)$ such that, for every finite abelian group G and every translation invariant projection P on $L_1(G)$ with $\|P\|=\lambda$, $d(P(L_1(G)),\ell_1^{\operatorname{rank}P})\le\varphi(\lambda)$?*

PROBLEM 6.2: *Does there exist a function $\varphi(\lambda)$ such that, for every n and every λ-doubly stochastic and a.l.m. projection P on ℓ_1^n, $d(P(\ell_1^n),\ell_1^{\operatorname{rank}P})\le\varphi(\lambda)$?*

PROBLEM 6.3: *Does there exist a function $\varphi(\lambda)$ such that for every n and every (λ-doubly stochastic orthogonal) projection P with $\|P\|=\lambda$ on ℓ_1^n there is a finite abelian group G and a translation invariant projection Q on $L_1(G)$ such that $\|Q\|\le\varphi(\lambda)$ and $d(P\ell_1^n,Q(L_1(G)))\le\varphi(\lambda)$?*

PROBLEM 6.4: *Does there exist a function $\varphi(\lambda)$ such that, for every n and every projection P with $\|P\| = \lambda$ on ℓ_1^n, there is an N and an orthogonal projection Q on ℓ_1^N such that $\|Q\|_0 \le \varphi(\lambda)$ and $d(P\ell_1^n, Q\ell_1^N) < \varphi(\lambda)$?*

Note that, by Proposition 1 of [J-J], for every projection P on $L_1(\mu)$ with $\|P\| = \lambda$ and every $\varepsilon > 0$ there is an $L_1(\nu)$ space and a surjective isometry $\varphi : L_1(\mu) \to L_1(\nu)$ such that, if $Q = \varphi P \varphi^{-1}$, then $\|Q\|_1 = \lambda$ and $\|Q\|_\infty \le \lambda(1+\varepsilon)$.

Remark 6.5: The argument used in the proof of Theorem 4.2 sheds more light on a.l.m. projections on ℓ_1^n and may serve as an alternative proof of Proposition 1.2 in the case $X = \ell_1^n$. Indeed, the argument yields the following:

THEOREM 6.6: *Let P be a projection on ℓ_1^n with $\|P\| = \lambda$. Then exactly one of the following statements holds:*

(a) *There is an operator $S \in L(\ell_1^n)$ satisfying $\mathrm{tr}SP = \lambda$, $\|S\|_\wedge = 1$ and $SP = PS$ (and hence P is a.l.m.).*

(b) *There exist positive numbers γ_0 and δ_0, depending on P, such that, for every $0 < \alpha < \delta_0$, the ball $B(P, \alpha + c\alpha^2)$ contains a projection Q with $\|Q\| \le \lambda - \frac{1}{2}\gamma_0\alpha + c\alpha^2$, where c is a universal constant.*

Indeed, we only have to replace $\Omega = \mathrm{Com}P + K$ by $\Omega = \mathrm{Com}P$ in the proof of Theorem 4.2 and ignore the Hermite property of all operators involved in the proof. If $S \in \Omega \cap \Delta$ then S satisfies the assumptions of Proposition 1.2 and hence P is a.l.m. On the other hand, if $\Omega \cap \Delta = \emptyset$ then there is a separating operator W, with $\mathrm{tr}(WT) = 0$ for all $T \in \Omega$ and $\mathbf{Re}(\mathrm{tr}WT) \ge \gamma_0 > 0$ for all $T \in \Delta$. The fact that $\mathbf{Re}(\mathrm{tr}(WT)) = 0$ for all $T \in \Omega$ means precisely, in this case, that $W = PW(I-P) + (I-P)WP$. The rest of the argument works perfectly well to show that if P is not a.l.m. then (b) holds. Note that the norm reduction we get in (b), for a projection P which is not a.l.m., is essentially **linear** in α, while in Proposition 1.2, we obtain a norm reduction of order α^2. This may seem to be a contradiction but it is not, because, in Proposition 1.2, the constant D is universal while in Theorem 6.6 the constants depend on P.

Remark 6.7: As mentioned in Remark 2.2 of [Z-1], we need a large constant $D = D(\lambda_0)$ in the definition of an a.l.m. projection in Theorem 5.2 above and in Theorem 2.4 of [Z-1]. Theorem 6.6 above shows that, in the special case of $X = \ell_1^n$, we are allowed to choose any positive D, independent on n and λ_0, in Definition 1.1 above and omit the restriction $\lambda < \lambda_0$. The choice $D = 1$ is natural, and, by Theorem 6.6, Proposition 1.2 remains valid with this or any other choice of $D > 0$.

PROBLEM 6.8: *Let P be an orthogonal projection on ℓ_1^n. Suppose that P is o.a.l.m. Is it a.l.m.?*

Theorem 6.6 is a first step towards a positive solution of Problem 6.8.

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